جامعة بيرزيت
BIRZEIT UNIVERSITY

Mathematics Department

Math 330
Test 2

2nd Semester 17-18

Student name: .... ID no.: ...... sec........

Q# 1) (25 points) (a) Solve the following system using Gaussian Elimination.

$$2x_1 + 4x_2 - 6x_3 = -4$$
$$x_1 + 5x_2 + 3x_3 = 10$$
$$x_1 + 3x_2 + 2x_3 = 5$$

(b) Find the cost of solving the above system

(c)Find the cost of solving nxn system using Gaussian Elimination.

(a)

$$\left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 1 & 5 & 3 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right] \quad m_{21} = \frac{1}{2} \quad m_{31} = \frac{1}{2}$$

4

$$\begin{array}{c} \\ R_2 - 0.5R_1 \\ R_3 - 0.5R_1 \end{array} \left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 0 & 3 & 6 & 12 \\ 0 & 1 & 5 & 7 \end{array}\right] \quad m_{32} = \frac{1}{3} = 0.3333$$

3

$$\begin{array}{c} \\ \\ R_3 - 0.333R_2 \end{array} \left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 0 & 3 & 6 & 12 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

$$\boxed{x_3 = \frac{3}{3} = 1}$$

$3x_2 + 6(1) = 12$

$3x_2 = 6 \qquad x_2 = \frac{12 - 6 \times 1}{3}$

$$\boxed{x_2 = 2}$$

3

$2x_1 + 4(2) - 6(1) = -4$

$2x_1 = -4 + 6 - 8 = -6 \Rightarrow \boxed{x_1 = -3}$

(b) Cost

1	3×2	2+3×2
2	2×1	1+2×1
	8	3+8

5

Backward:

$1 + 3 + 5 = 9$

Total

$9 + 9 = 28$

1

Total cost

1	$(n-1)(n)$	$n-1 + (n-1)n$
2	$(n-2)(n-1)$	$n-2 + (n-2)n-1$
k	$(n-k)(n-k+1)$	$(n-k)(n-k+1)$
n-1		

5

Cost $$\sum_{n=1}^{n-1} (n-k)(n-k+1) + \sum_{n=1}^{n-1} (n-k) + (n-k)(n-k+1) + n^2$$

$$= 2\left[\sum (n-k)^2 + \sum (n-k)\right] + \sum (n-k) +$$

$$= 2\sum_{n=1}^{n-1} (n-k)^2 + 3\sum_{n=1}^{n-1} (n-k) + n^2$$

5

$$= 2\sum^{n-1} p^2 + 3\sum p + n^2$$

$$= 2\left(\frac{n(n-1)(2n-1)}{6}\right) + 3\left(\frac{(n-1)(n)}{2}\right) + n^2$$

$$= \frac{4n^3 + 9n^2 - 7n}{6}$$

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Total $9 + 9 = 28$

Total cost

1	$(n-1)(n)$	$n-1+(n-1)n$
2	$(n-2)(n-1)$	$n-2+(n-2)n-1$
k	$(n-k)(n-k+1)$	$(n-k)(n-k+1)$
n-1		

<u>Cost</u> $$\sum_{n=1}^{n-1}(n-k)(n-k+1) + \sum_{n=1}^{n-1}(n-k)+(n-k)(n-k+1) + n^2$$

$$= 2\left[\sum(n-k)^2 + \sum(n-k)\right] + \sum(n-k) +$$

$$= 2\sum_{n=1}^{n-1}(n-k)^2 + 3\sum_{n=1}^{n-1}(n-k) + n^2$$

$$= 2\sum^{n-1} p^2 + 3\sum p + n^2$$

$$= 2\left(\frac{n(n-1)(2n-1)}{6}\right) + 3\left(\frac{(n-1)(n)}{2}\right) + n^2$$

$$= \frac{4n^3+9n^2-7n}{6}$$

Q#2)(25 points) (a) Consider the data

$$(1, f(1), (1.2, f(1.2), (1.3, f(1.3)) \qquad \text{Where } f(x) = \frac{1}{x+1}$$

Find Lagrange interpolation polynomial $p_2(x)$ and use it to estimate $f(1.25)$

(b) Find Newton interpolation polynomial $p_3(x)$ for $x \in [1,2.2]$ with equally spaced nodes x_0, x_1, x_2, x_3 and use it to estimate $f(1.25)$

(c) Find the best upper bound for $E_3(x)$ above

(3) $P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$

$= \frac{(x-1.2)(x-1.3)}{(-0.2)(-0.3)} y_0 + \frac{(x-1)(x-1.3)}{(0.2)(-0.1)} y_1 + \frac{(x-1)(x-1.2)}{(0.3)(0.1)} y_2$

(4)

$= 8.333(x-1.2)(x-1.3) - 22.73(x-1)(x-1.3) + 14.49(x-1)(x-1.2)$

(8) $P_2(1.25) = 0.4444 \quad (-0.02083 + 0.2841 + 0.1811)$

x_k	$f(x_k)$	1st D.D	2nd D.D	3rd D.D
1	0.5			
1.4	0.4167	-0.2083		
1.8	0.3571	-0.149	0.074125	
2.2	0.3125	-0.1115	0.04688	-0.0227

(6)

$P_2(x) = 0.5 - 0.2083(x-1) + 0.07413(x-1)(x-1.4)$
$\qquad - 0.0227(x-1)(x-1.4)(x-1.8)$

(2)

$P_2(1.25) \simeq 0.4445$ (2)

(c) $|E_3(x)| \le \frac{h^4 M_4}{24}$ (1), $M_4 = 0.75$ (2), $h = 0.4$ (1)

$\le \frac{(0.4)^4 \cdot 0.75}{24} = 0.008$ (1)

(e) $y = \frac{c}{x} + D\cos x$

$$\frac{y}{\cos x} = \frac{c}{x\cos x} + D$$

$$\frac{y}{\cos x} = c\left(\frac{1}{x\cos x}\right) + D$$

$$Y = AX + B$$

$$\left.\begin{aligned} X &= \frac{1}{x\cos x} \\ Y &= \frac{y}{\cos x} \end{aligned}\right\} \quad A, B$$

$c = A$, $D = B$

or $\quad \underbrace{xy}_{Y} = D\,x\cos x + C$

$$Y = AX + B$$

5

Q#3)(25 points) Consider the data

$$(1,3.62), (1.2,2.754), (1.4,2.342)$$

(a) Derive the normal equations for the best fit of the form $f(x) = \frac{A}{x} + B\cos x$

(b) Find A, B using the normal equations derived above and given data.

(c) Find a suitable Linearization for $f(x) = \frac{C}{x} + D\cos x$ (Don't find C, D)

$$E(A,B) = \sum_{k=1}^{n} \left(\frac{A}{x_k} + B\cos x_k - y_k\right)^2$$

4

$$\frac{\partial E}{\partial A} = 0 = 2\sum_{k=1}^{n}\left(\frac{A}{x_k} + B\cos x_k - y_k\right)\cdot\frac{1}{x_k}$$

4

$$A\sum_{k=1}^{n}\frac{1}{x_k^2} + B\sum_{k=1}^{n}\frac{\cos x_k}{x_k} = \sum_{k=1}^{n}\frac{y_k}{x_k} \quad --(1)$$

4

$$\frac{\partial E}{\partial B} = 0 = 2\sum\left(\frac{A}{x_k} + B\cos x_k - y_k\right)\cdot\cos x_k$$

4

$$A\sum_{k=1}^{n}\frac{\cos x_k}{x_k} + B\sum_{k=1}^{n}(\cos x_k)^2 = \sum y_k\cos x_k \quad --(2)$$

8

x_k	y_k	$\frac{y_k}{x_k}$	$\cos x_k$	$\frac{\cos x_k}{x_k}$	$y_k\cos x_k$	$\frac{1}{x_k^2}$	$(\cos x_k)^2$
1	3.62	3.62	0.5403	0.5403	1.956	1	0.2919
1.2	2.754	2.295	0.3624	0.302	0.9980	0.6944	0.1313
1.4	2.342	1.673	0.1670	0.1214	0.3981	0.5102	0.02889
Tot		7.588		0.9637	3.3519	2.2046	0.4521

$$2.2046A + 0.9637B = 7.588$$

$$0.9637A + 0.4521B = 3.3519$$

4

$$A = \frac{\begin{vmatrix} 7.588 & 0.9637 \\ 3.3519 & 0.4521 \end{vmatrix}}{\begin{vmatrix} 2.2046 & 0.9637 \\ 0.9637 & 0.4521 \end{vmatrix}} = \frac{0.2003}{0.06798} \simeq 2.946$$

$$B = \frac{\begin{vmatrix} 2.2046 & 7.588 \\ 0.9637 & 3.3519 \end{vmatrix}}{0.06798} \simeq 1.13$$

Q#4) (25 points) a) Derive the following formula and **its error** using **Lagrange** interpolating Polynomial

$$f'(x_0) \cong \frac{-3f_0 \; +4f_1 \; -f_2}{2h} + \frac{h^2}{3} f'''(c) \text{ -}$$

b) Use the above formula to estimate f' (1.1) for the data (1.1,0.4238),(1.2, 1.003), (1.3,1.662).

c) Find the optimal h for the above formula.

3 $$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

3 $$P_2'(x) = \frac{(x-x_1)+(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)+(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)+(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$P_2'(x_0) = \frac{(x_0-x_1)+(x_0-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x_0-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x_0-x_2)}{(x_2-x_0)(x_2-x_1)} y_2$$

utting $x_0 = x$, $x_1 = x+h$, $x_2 = x+2h$

3 $$P_2'(x) = \frac{(-h)+(-2h)}{(-h)(-2h)} y_0 + \frac{(-2h)}{(+h)(-h)} y_1 + \frac{(-2h)}{(2h)(h)} y_2$$

1 $$= \frac{-3y_0}{2h} + \frac{4y_1}{2h} - \frac{y_2}{2h} = \frac{-3f(x)+4f(x+h)-f(x+2h)}{2h}$$

1 $$E_2(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{3!} f^{(3)}(c)$$

3 $$E'(x) = (x-x_0)(x-x_1) + (x-x_0)(x-x_2) + (x-x_1)(x-x_2) \frac{f^{(3)}(c)}{3!}$$

$$E'(x_0) = \frac{(x_0-x_1)(x_0-x_2)}{6} f^{(3)}(c)$$

2 $$= \frac{(-h)(-2h)}{6} f^{(3)}(c) = \frac{h^2}{3} f^{(3)}(c)$$

$$f'(1.1) = \frac{-3f(1.1)+4f(1.2)-f(1.3)}{2h}$$ 2

$$= \frac{-3(0.4238)+4(1.003)-1.662}{2(0.1)}$$

2

$$= 5.393$$

$$g(h) = \frac{8\epsilon}{2h} + \frac{M_3 h^2}{3}$$ 2

$$g'(h) = \frac{-4\epsilon}{h^2} + \frac{2M_3 h}{3} = 0$$ 2

$$\frac{4\epsilon}{h^2} = \frac{2M_3 h}{3}$$

1

$$\frac{12\epsilon}{2M_3} = h^3 \Rightarrow \boxed{h = \left(\frac{6\epsilon}{M_3}\right)^{1/3}}$$